Stability and control of stochastic nonlinear delay systems

Quanxin Zhu

Hunan Normal University

A presentation at The 18th International Workshop on Markov Processes and Related Topics

July 31, 2023

Outline

Stochastic nonlinear system:

Consider the following stochastic differential equation:

$$dx(t) = f(x(t))dt + g(x(t))dw(t), \ \ t \geq t_0 \geq 0, \qquad (1)$$

with the initial value $x(t_0) = x_0 \in \mathbb{R}^d$, where f and g are two measurable functions, and $w(t)$ is a Brownian motion.

沃尔夫奖、高斯奖和京都奖 Ito Kiyoshi (1915-2008)

Important applications:

The 1997 Nobel Prize in economics, Black-Scholes formula:

$$dx(t) = \mu x(t)dt + \sigma x(t)dw(t), \ t \geq t_0 \geq 0.$$

Myron Scholes (1941-) Robert C. Merton(1944-)

Common questions:

- If the solution $x(t)$ of (1) exists, then it is a Markov process.

- We are concerned with the existence and uniqueness of solution to system (1).

- We are concerned with the long time behavior of the solution $x(t)$:

$$(i) \quad \lim_{t \longrightarrow \infty} x(t) = ? \quad a.s.$$

$$(ii) \quad \lim_{t \longrightarrow \infty} \mathbf{E}|x(t)|^p = ?$$

Various complex phenomena lead to system instability

- In practical systems, it often happens that some stochastic differential equations (SDEs) are unstable.

Earthquake

Tsunami

War

Financial Crisis

Our concerning question

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \geq t_0. \tag{2}$$

Question 1: Under what condition the SDE (2) is stable?

$$dx(t) = [f(x(t), t) + u(x(t))]dt + g(x(t), t)dB(t), \quad t \geq t_0. \tag{3}$$

Question 2: If the SDE (2) is unstable, then whether there exists a control $u(x(t))$ such that the SDE (3) is stable ?

- Naturally, an interesting and challenging problem is how to design a control function to guarantee the stability of controlled SDEs when the original system is unstable?

Existing methods

- Continuous-time feedback control

For stabilization of SDEs, many results have been presented via continuous-time feedback control, we can refer to the following papers.

- X. Mao, Y. G. George, C. Yuan, Stabilization and destabilization of hybrid systems of stochastic differential equations, Automatica, 43(2007) 264-273.
- F. Deng, Q. Luo, X. Mao, Stochastic stabilization of hybrid differential equations, Automatica, 48(2012)2321-2328.
- Q. Zhu, H. Wang, Output feedback stabilization of stochastic feedforward systems with unknown control coefficients and unknown output function, Automatica, 87(2018)166-175.
- H. Wang, Q. Zhu, Global stabilization of a class of stochastic nonlinear time-delay systems with SISS inverse dynamics, IEEE Tran. Automa. Control 65(10)(2020)4448-4455.

Existing methods

- Sampled-data control

For stabilization of SDEs, many results have been presented via sampled-data control, we can refer to the following papers.

- X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, Automatica, 49(12)(2013)3677-3681.
- X. Mao, W. Liu, L. Hu, Q. Luo, J. Lu, Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations, Syst. Control Lett., 73(2014)88-95.
- Q. Zhu, Q. Zhang, pth moment exponential stabilisation of hybrid stochastic differential equations by feedback controls based on discrete-time state observations with a time delay, IET Control Theory Appl., 11(2017)1992-2003.
- G. Song, B. Zeng, Q. Luo, X. Mao, Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time observations of state and mode, IET Control Theory Appl., 11(2017)301-307.

Disadvantages of the above methods

- The continuous-time state feedback control requires that the controller observes the state of the process and makes a decision every time.
- Obviously, this is too expensive and not realistic in real lives.
- Sampled-data control requires that the controller observes the state of the process and makes a decision every time according to the fix time.
- Obviously, it does not take into account the system behavior.

Our method–the event-triggered control

- The event-triggered control is a better sampled-data control.
- The sampling and the updating of the controller are triggered by the occurrence of certain events depending on the system state.
- The event-triggered control is more effective in the real control problem.
- For deterministic systems, there are a large number of results.
- For stochastic systems, there are some results but most results are concentrated on the discrete-time systems.

The event-triggered control problem for deterministic systems

- V. Dolk, M. Heemels, Event-triggered control systems under packet losses, Automatica,80(2017)143-155.
- A. Selivanov, E. Fridman, Distributed event-triggered control of diffusion semilinear PDEs, Automatica, 68(2016)344-351.
- K. Hashimoto, S. Adachi, D. V. Dimarogonas, Event-triggered intermittent sampling for nonlinear model predictive control, Automatica, 81(2017)148-155.
- D.P. Borgers, W. Heemels, Event-separation properties of event-triggered control systems, IEEE Trans. Autom. Control, 59(10)(2014)2644-2656.

The event-triggered control problem for discrete-time stochastic systems

- D. Ding, Z. Wang, B. Shen, and G. Wei, Event-triggered consensus control for discrete-time stochastic multi-agent systems: the input to state stability in probability, Automatica, 62(2015)284-291.
- D. Quevedo, V. Gupta, W. Ma, S. Yuksel, Stochastic stability of event-triggered anytime control, IEEE Trans. Autom. Control, 59(12)(2014)3373-3379.
- T. Zhang, F. Deng, P. Shi, Event-triggered H-infinity filtering for nonlinear discrete-time stochastic systems with application to vehicle roll stability control, International Journal of Roubust Nonlinear Control, 30(2020)8430-8448.
- W. Xie, Q. Zhu, Stability of discrete-time stochastic nonlinear systems with event-triggered state-feedback control, Physics A, 547(2020)123823.

The event-triggered control problem for continuous-time stochastic systems

- L. Wu, Y. Gao, J. Liu, H. Li, Event-triggered sliding mode control of stochastic systems via output feedback, Automatica, 82(2017)79-92.
- R. Anderson, D. Milutinovic, D. Dimarogonas, Self-triggered sampling for second-moment stability of state-feedback controlled SDE systems, Automatica, 54(2015)8-15.

However, delays are ignored in the above works even if delays are a major source for causing instability and poor performances.

Open problem: How to solve the event-triggered control problem for continuous-time stochastic delay systems?

Notations

- $\mathcal{Z}_+ = \{0, 1, 2, 3, ...\}$ and $\mathbb{R}_+ = [0, +\infty)$.
- $\mathcal{L}_\infty^n$ denotes the class of measurable and essentially bounded functions υ from $\mathbb{R}_+$ to $\mathbb{R}^n$ with $||\upsilon||_\infty = \mathrm{ess\,sup}_{t\geq 0} \mid \upsilon(t) \mid < \infty$.
- $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the uniform norm $\|\phi\|_\tau = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$.
- $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$ measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic variables $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0} \mathbf{E}|\phi(\theta)|^2 < \infty$.

Our model

We are concerned with the following Itô stochastic nonlinear delay system with exogenous disturbances:

$$dx(t) = [Ax(t) + Bx(t-\tau) + f(t, x(t)) + g(t, x(t-\tau)) + Cu(t) + \upsilon(t)]dt + \sigma(t, x(t), x(t-\tau))dW(t), \quad (4)$$

where the initial data $x_0 = \phi = \{\phi(\theta), -\tau \leq \theta \leq 0\} \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $\upsilon(t) \in \mathcal{L}^n_\infty$ are state vector, feedback control vector, and unknown exogenous disturbance input vector, respectively. A, B, C are constant matrices with compatible dimensions.

Existence-uniqueness condition of solution

- f, g, σ are assumed to satisfy the global Lipschitz condition: there exist four nonnegative constants l_1, l_2, r_1, r_2 such that

$$|f(t,x_1) - f(t,x_2)|^2 \leq l_1|x_1 - x_2|^2,$$
$$|g(t,x_1) - g(t,x_2)|^2 \leq l_2|x_1 - x_2|^2,$$
$$|\sigma(t,x_1,x_3) - \sigma(t,x_2,x_4)|^2 \leq r_1|x_1 - x_2|^2 + r_2|x_3 - x_4|^2,$$

where $t \in \mathbb{R}_+, x_1, x_2, x_3, x_4 \in \mathbb{R}^n$.

- Moreover, $f(t,0) = 0$, $g(t,0) = 0$, $\sigma(t,0,0) = 0$.

A new event-triggered scheme

The sampling time sequence $\{t_i : i \in \mathcal{Z}_+\}$ satisfies $t_0 = 0$ and

$$t_{i+1} = \inf\{t : t > t_i, h(t) > 0\},$$

where $h(t)$ is an event-generator function to be determined later. The controller is taken as follows:

$$u(t) = Kx(t_i), \quad t \in [t_i, t_{i+1}), \; i \in \mathcal{Z}_+, \tag{5}$$

where $K \in \mathbb{R}^{m\times n}$ is the feedback gain to be determined later. Then, system (4) can be rewritten as

$$\begin{aligned} dx(t) = {} & [Ax(t) + Bx(t-\tau) + f(t, x(t)) + g(t, x(t-\tau)) \\ & + CKx(t_i) + \upsilon(t)]dt + \sigma(t, x(t), x(t-\tau))dW(t), \end{aligned} \tag{6}$$

for $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_+$.

A new event-triggered scheme

Let $\varepsilon(t)$ denote the measurement error between the sampled state and current state. Then, we have

$$\varepsilon(t) = x(t_i) - x(t). \tag{7}$$

By (7), we can rewrite system (6) as follows:

$$\begin{aligned} dx(t) = [(A + CK)x(t) + Bx(t-\tau) + f(t, x(t)) \\ +g(t, x(t-\tau)) + CK\varepsilon(t) + \upsilon(t)]dt \\ +\sigma(t, x(t), x(t-\tau))dW(t). \end{aligned} \tag{8}$$

In this paper, we consider an event-generator function h as follows:

$$h(t) = |\varepsilon(t)|^2 - \eta_1|x(t_i)|^2 - \eta_2, \;\; t \in [t_i, t_{i+1}), \; i \in \mathcal{Z}_+, \tag{9}$$

where $\eta_1, \eta_2 \in \mathbb{R}_+$ are the weight parameter and threshold parameter to be designed, and they satisfy $\eta_1^2 + \eta_2^2 \neq 0$.

Definition of input-to-state practically exponentially mean-square stability

- System (8) is said to be input-to-state practically exponentially mean-square stable with respective to the exogenous disturbance input $\upsilon(t)$ if there exist three constants $\alpha > 0$, $\beta > 0$, $d \geq 0$ and a function $\gamma \in \mathcal{H}$ such that

$$\mathbf{E}|x(t;\phi)|^2 \leq \alpha e^{-\beta t} \sup_{-\tau \leq \theta \leq 0} \mathbf{E}|\phi(\theta)|^2 + \gamma(|\upsilon|_\infty) + d$$

  for $t \in \mathbb{R}_+, \phi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $\upsilon \in \mathcal{L}^n_\infty$.
- Especially, when $d = 0$, system (4) is said to be input-to-state exponentially mean-square stable.
- Furthermore, when $d = 0$ and $\upsilon = 0$, system (4) is said to be exponentially mean-square stable.

Definition of input-to-state practically exponentially mean-square stabilization

- System (4) is said to be input-to-state practically exponentially mean-square stabilizable if there exist a feedback gain matrix K and triggering parameters η_1, η_2 such that system (8) is input-to-state practically exponentially mean-square stable with respective to the exogenous disturbance input $\upsilon(t)$.
- Especially, when $d = 0$, system (4) is said to be input-to-state exponentially mean-square stabilizable.
- Furthermore, when $d = 0$ and $\upsilon = 0$, system (4) is said to be exponentially mean-square stabilizable.

Theorem 1

Let the matrix $K \in \mathbb{R}^{m\times n}$ and two $\eta_1, \eta_2 \in \mathbb{R}_+$ satisfy $0 \le \eta_1 < \frac{1}{2}, \eta_2 \ge 0$. If there exist positive definite matrices $P, Q, G_i (i = 1, 2, 3, 4)$ such that the following matrix inequality holds:

$$\Pi = \begin{pmatrix} \pi_{11} & PB \\ \star & \pi_{22} \end{pmatrix} < 0,$$

where

$$\begin{aligned} \pi_{11} = {} & PA + PCK + A^T P + K^T C^T P \\ & + PG_1^{-1} P + l_1 G_1 + PG_2^{-1} P \\ & + PG_3^{-1} P + PCKG_4^{-1} K^T C^T P \\ & + \lambda_{\max}(P) r_1 + Q + \lambda_{\max}(G_4) \frac{2\eta_1}{1 - 2\eta_1}, \end{aligned}$$

$$\pi_{22} = -Q + \lambda_{\max}(P) r_2 + l_2 G_2.$$

Then, system (8) is input-to-state practically exponentially mean-square stable with respect to $\upsilon(t)$.

- Zhu Quanxin, Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control, *IEEE Transactions on Automatic Control*, 64(9)(2019) 3764-3771.

Corollary 1

Let all the conditions of Theorem 1 hold. When $\eta_2 = 0$ in $h(t)$, system (8) is input-to-state exponentially mean-square stable with respect to $v(t)$. Furthermore, when $v(t) = 0$ in (8) and $\eta_2 = 0$ in $h(t)$, system (8) is exponentially mean-square stable.

Theorem 2

Let two $\eta_1, \eta_2 \in \mathbb{R}_+$ satisfy $0 \leq \eta_1 < \frac{1}{2}, \eta_2 \geq 0$. System (4) is input-to-state practically exponentially mean-square stabilizable if there exist positive definite matrices $P, Q, G_i (i = 1, 2, 3, 4)$ and a constant matrix $Y \in \mathbb{R}^{m \times n}$ such that

$$\Pi = \begin{pmatrix} \Lambda & X \\ \star & \pi_{22} \end{pmatrix} < 0, \qquad (10)$$

where

$$\Lambda = \begin{pmatrix} \pi_{11} & P & P & P & CY \\ \star & -G_1 & 0 & 0 & 0 \\ \star & \star & -\hat{G}_2 & 0 & 0 \\ \star & \star & \star & -G_3 & 0 \\ \star & \star & \star & \star & -G_4 \end{pmatrix},$$

$$
\begin{aligned}
& X = [PB \quad 0 \quad 0 \quad 0 \quad 0]^T, \\
& \pi_{11} = PA + CY + A^T P + Y^T C^T + l_1 G_1 \\
& \qquad + \lambda_{\max}(P) r_1 + Q + \lambda_{\max}(G_4) \frac{2\eta_1}{1 - 2\eta_1}, \\
& \pi_{22} = -Q + \lambda_{\max}(P) r_2 + l_2 G_2.
\end{aligned}
$$

Furthermore, the gain matrix K of the desired feedback controller (5) is designed by

$$K = YP^{-1}. \tag{11}$$

Proof of Theorems

I do not present the proof since it is complex and tedious. Instead, I only mention some techniques as follows:

- Construct the following Lyapunov-Krasovskii functional:

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qx(s)ds.$$

- By using the event-triggered condition and $\varepsilon(t)$, how to deal with $x(t_k)$?
- Stochastic analysis, the Dynkin formula and some inequalities techniques, etc.
- Since t_i is a stopping time,

$$\mathbf{E}\int_{t_i}^{t} \mathcal{L}|\varepsilon(s)|^2 ds = \int_{t_i}^{t} \mathbf{E}\mathcal{L}|\varepsilon(s)|^2 ds$$

does not hold.

Remark

- In the proof of Theorems 1 and 2, we obtain

$$\mathbf{E}|x(t)|^2 \leq \alpha e^{-\beta t} \sup_{-\tau\leq\theta\leq 0} \mathbf{E}|\xi(\theta)|^2 + \gamma(|v|_\infty) + d, \qquad (12)$$

where

$$\alpha := \frac{1}{\lambda_{\min}(P)}[\lambda_{\max}(P) + \lambda_{\max}(Q)\beta\tau^2 e^{\beta\tau}],$$

$$\gamma(s) := \frac{\lambda_{\max}(G_3)|v(s)|^2}{\lambda_{\min}(P)\beta},$$

$$d := \frac{\eta_2\lambda_{\max}(G_4)}{(1-2\eta_1)\lambda_{\min}(P)\beta}.$$

Remark

- Thus, we provide the ultimately bounded estimation for the state $x(t)$ and the ultimate bound can be determined by (12), which depends on the triggered parameter η_2 and the infinite norm $|\upsilon|_\infty$.
- In particular, when $\eta_2 = 0$ and $\upsilon = 0$, the system state $x(t)$ converges to zero with the exponential decay rate β, which is determined by the equation $\beta\lambda_{\max}(P) + \lambda_{\max}(Q)\beta\tau e^{\beta\tau} = \gamma$.

Theorem 3

Let all the conditions in Theorem 1 hold. Then, there is a positive constant T^* such that $t_{i+1} - t_i \geq T^*$ for all $i \in \mathcal{Z}_+$.

Remark

- In Theorem 3, we obtain the lower bounds of inter-execution times based on the proposed event-triggered control method: $t_{i+1} - t_i \geq T^* > 0$ for all $i \in \mathcal{Z}_+$.
- This fact implies that the Zeno behavior does not happen in our proposed event-triggered control scheme but we can still ensure the input-to-state exponential mean-square stability of system (8).
- Thus, our result is quite different from the traditional event-triggered control results established on the Zeno behaviors [1] –[4].
- Moreover, noise disturbance was ignored in [1] –[4].

References on the traditional event-triggered control

1. W. Zhu, Z. Jiang, Event-based leader-following consensus of multi- agent systems with input time delay, IEEE Trans. Autom. Control, 60(5)(2015)1362-1367.
2. C. Persis, R. Sailer, F. Wirth, Parsimonious event-triggered distributed control: a Zeno free approach, Automatica, 49(7)(2013)2116-2124.
3. J. Lunze, D. Lehmann, A state-feedback approach to event-based control, Automatica, 46(1)(2010) 211-215.
4. D. Dimarogonas, E. Frazzoli, K. Johansson, Distributed eventtrig- gered control for multi-agent systems, IEEE Trans. Autom. Control, 57(5)(2012) 1291-1297.

Remark

In the proof of Theorem 3, we obtain

$$t_{i+1} - t_i \geq \frac{1}{a_1} \ln(1 + \frac{a_1(\eta_1 \mathbf{E}|x(t_i)|^2) + \eta_2}{a_2 \mathbf{E}|x(t_i)|^2 + \bar{K}} > 0. \quad (13)$$

- (13) gives a rough prediction for the next triggering time t_{i+1} by the computation method.
- Moreover, the execution interval will disappear when η_1 and η_2 go to zero. Thus, we always choose η_1 and η_2 to control the event-triggered frequency.

H_∞ control of stochastic networked control systems with time-varying delays

We are concerned with the following Itô stochastic nonlinear delay system with exogenous disturbances:

$$\begin{aligned} dx(t) &= [A(t)x(t) + B(t)x(t-\tau(t)) + C(t)u(t) + F_\upsilon \upsilon(t)]dt \\ &\qquad +[G_1 x(t) + G_2 x(t-\tau(t))]dW(t), \\ y(t) &= Dx(t) + Eu(t), \end{aligned} \tag{14}$$

where the initial data $x_0 = \xi = \{\xi(\theta), -\tau \leq \theta \leq 0\} \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\upsilon(t) \in \mathcal{L}_2[0, \infty)$ and $y(t) \in \mathbb{R}^p$ are state vector, control input vector, disturbance input vector, and controlled out vector, respectively. $\tau(t)$ is the time-varying delay, which is differential and satisfies $0 \leq \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \rho < 1$.

H_∞ control of stochastic networked control systems with time-varying delays

$A(t) = A + \triangle A(t)$, $B(t) = B + \triangle B(t)$, $C(t) = C + \triangle C(t)$, $A, B, C, D, E, F_\upsilon$ and G are constant matrices with appropriate dimensions; $\triangle A(t), \triangle B(t)$ and $\triangle C(t)$ denote the time-varying parameter uncertainties such that $[\triangle A(t), \triangle B(t), \triangle C(t)] = MF(t)[N_1, N_2, N_3]$, where M and $N_i (i = 1, 2, 3)$ are the known constant matrices and $F(t)$ satisfies: $F^T(t)F(t) \leq I$.

We now introduce the following event-triggered scheme. There is a sampling time sequence $\{t_i : i \in \mathcal{Z}_o^+\}$ such that $t_0 = 0$ and

$$t_{i+1} = \inf\{t : t > t_i, J(t) > 0\},$$

where $J(t)$ is defined in (19) below and it is usually called the event-generator function.

H_∞ control of stochastic networked control systems with time-varying delays

The controller is defined as

$$u(t) = Kx(t_i), \quad t \in [t_i, t_{i+1}), \ i \in \mathcal{Z}_o^+, \tag{15}$$

where $K \in \mathbb{R}^{m \times n}$ is the feedback matrix. Usually, such a state feedback sampled-data control is ZOH.
Obviously, we can rewrite system (14) as

$$\begin{aligned} dx(t) &= [A(t)x(t) + B(t)x(t-\tau(t)) \\ &\quad + C(t)Kx(t_i) + F_\upsilon \upsilon(t)]dt \\ &\quad + [G_1 x(t) + G_2 x(t-\tau(t))]dW(t), \\ y(t) &= Dx(t) + EKx(t_i), \end{aligned} \tag{16}$$

for $t \in [t_i, t_{i+1}), \ i \in \mathcal{Z}_o^+$.

H_∞ control of stochastic networked control systems with time-varying delays

It is clear that there exists a unique solution for systems (14) and (16). As a usual, we use $x(t;\xi)$ to denote the solution of system (14) and $y(t;\xi)$ to denote the solution of system (16) for the initial data $x_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. Next, we define the following measurement error:

$$e(t) = x(t_i) - x(t). \tag{17}$$

Then, it follows from (16) and (17) that

$$\begin{aligned} dx(t) &= [(A(t) + C(t)K)x(t) + B(t)x(t - \tau(t)) \\ &\quad + C(t)Ke(t) + F_\upsilon \upsilon(t)]dt \\ &\quad + [G_1 x(t) + G_2 x(t - \tau(t))]dW(t), \\ y(t) &= (D + EK)x(t) + EKe(t). \end{aligned} \tag{18}$$

H_∞ control of stochastic networked control systems with time-varying delays

The event-generator function J under our research is given as

$$J(t) = \lambda_1 |e(t)|^2 - \lambda_2 |x(t_i)|^2, \ t \in [t_i, t_{i+1}), \ i \in \mathcal{Z}_o^+, \qquad (19)$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two parameters.

Two stability definitions

Definition

System (18) is called robustly exponentially stable in mean-square (REsMS) if system (18) with $\upsilon(t) = 0$ is exponentially stable in mean-square for all admissible uncertainties $\triangle A(t)$, $\triangle B(t)$, and $\triangle C(t)$, i.e., there are two constants $\alpha > 0$ and $\beta > 0$ satisfying

$$\mathbb{E}|x(t;\xi)|^2 \leq \alpha e^{-\beta t} \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2.$$

Definition

System (18) is called REsMS with an H_∞ disturbance attenuation level γ if it is REsMS and under the zero initial condition,

$$\mathbb{E}||y(t;\xi)||_2 \leq \gamma||\upsilon(t)||_2$$

for any nonzero $\upsilon(t) \in \mathcal{L}_2[0,\infty)$.

Theorem 4

Given $K \in \mathbb{R}^{m\times n}$ and $\lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 > 2\lambda_2$, system (18) is robustly exponentially stable in mean-square with an H_∞ disturbance attenuation level γ , if there are matrices $P > 0$, $Q > 0$ and $R > 0$ satisfying the matrix inequality as follows:

$$\Pi = \begin{pmatrix} \Lambda_{11} & PB & PF_\upsilon \\ \star & \Lambda_{22} & 0 \\ \star & \star & -\gamma^2 I \end{pmatrix} < 0, \tag{20}$$

where

$$\begin{aligned} \Lambda_{11} = {} & PA + A^T P + PCK + K^T C^T P + 4PMM^T P \\ & + N_1^T N_1 + K^T N_3^T N_3 K + PCC^T P + 2G_1^T PG_1 \\ & + Q + 2(D + EK)^T (D + EK) + \tau R \\ & + [\lambda_{\max}(K^T N_3^T N_3 K) + 2\lambda_{\max}(K^T E^T EK) \\ & + \lambda_{\max}(K^T K)] \frac{2\lambda_2}{\lambda_1 - 2\lambda_2} I, \end{aligned}$$

$$\Lambda_{22} = -(1-\rho)Q + N_2^T N_2 + 2G_2^T P G_2.$$

- Zhu Quanxin*, Huang Tingwen, H_∞ control of stochastic networked control systems with time-varying delays: The event-triggered sampling case, *International Journal of Robust and Nonlinear Control* 31(18)(2021)9767-9781.

Theorem 5

Given $K \in \mathbb{R}^{m\times n}$ and $\lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 > 2\lambda_2$, system (8) is robustly exponentially mean-square stable with an H_∞ disturbance attenuation level γ , if there are matrices $P > 0$, $Q > 0$ and $R > 0$ satisfying the LMI as follows:

$$\Pi = \begin{pmatrix} \tilde{\Lambda}_{11} & PB & PCK & PF_\upsilon & PM & PC \\ \star & \tilde{\Lambda}_{22} & 0 & 0 & 0 & 0 \\ \star & \star & \tilde{\Lambda}_{33} & 0 & 0 & 0 \\ \star & \star & \star & -\gamma^2 I & 0 & 0 \\ \star & \star & \star & \star & -\frac{I}{4} & 0 \\ \star & \star & \star & \star & \star & -I \end{pmatrix} < 0, \qquad (21)$$

where

$$\tilde{\Lambda}_{11} = PA + A^T P + PCK + K^T C^T P + N_1^T N_1 + K^T N_3^T N_3 K \\ +2G_1^T PG_1 + Q + 2(D + EK)^T (D + EK) + \tau R + 2\lambda_2 I,$$

$$\tilde{\Lambda}_{22} = -(1-\rho)Q + N_2^T N_2 + 2G_2 P G_2,$$
$$\tilde{\Lambda}_{33} = -(\lambda_1 - 2\lambda_2)I + K^T N_3^T N_3 K + 2K^T E^T E K.$$

- Zhu Quanxin*, Huang Tingwen, H_∞ control of stochastic networked control systems with time-varying delays: The event-triggered sampling case, *International Journal of Robust and Nonlinear Control* 31(18)(2021)9767-9781.

Proof of Theorems

I do not present the proof since it is complex and tedious. Instead, I only mention some techniques as follows:

- Construct the following Lyapunov-Krasovskii functional:

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Qx(s)ds + \int_{-\tau}^{0} d\theta \int_{t+\theta}^{t} x^T(s)Rx(s)ds.$$

- **Step 1:** We will prove that system (18) with $\upsilon(t) = 0$ is REsMS for all admissible uncertainties $\triangle A(t), \triangle B(t)$ and $\triangle C(t)$.
- **Step 2:** We will prove the following fact:

$$\mathbb{E}||y(t;\xi)||_2 \leq \gamma||\upsilon(t)||_2.$$

- Stochastic analysis, the Dynkin formula, Fubini's theorem and some inequalities techniques, etc.

Theorem 6

For given positive constants $\lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 > 2\lambda_2$ and positive constants $\varepsilon > 0$, $\rho > 0$. System (18) is robustly exponentially mean-square stable with an H_∞ disturbance attenuation level γ, if there exist positive definite matrices X, $\bar{Q}$, and matrix $\bar{K}$ satisfying the LMI as follows:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{bmatrix} < 0, \qquad (22)$$

where

$$\Xi_{11} = \begin{bmatrix} \tilde{A}_{11} & B\bar{Q} & C\bar{K} & F_v & M & C \\ * & \tilde{A}_{22} & 0 & 0 & 0 & 0 \\ * & * & \tilde{A}_{33} & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -\frac{I}{4} & 0 \\ * & * & * & * & * & -I \end{bmatrix}$$

$$\Xi_{12} = \begin{bmatrix} XN_1^{\mathrm{T}} & \bar{K}^{\mathrm{T}}N_3^{\mathrm{T}} & X^{\mathrm{T}}G_1 & X & (DX + E\bar{K})^{\mathrm{T}} & \sqrt{\bar{\tau}}X \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$$\Xi_{13} = \begin{bmatrix} X & 0 & 0 & X & 0 & 0 \\ * & \bar{Q}^{\mathrm{T}}N_2^{\mathrm{T}} & \bar{Q}G_2 & 0 & 0 & 0 \\ * & * & 0 & \frac{\varepsilon}{\sqrt{\lambda_1 - 2\lambda_2}} & \bar{K}^{\mathrm{T}}N_3^{\mathrm{T}} & \bar{K}^{\mathrm{T}}E^{\mathrm{T}} \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Xi_{22} = \mathrm{diag}\left\{ -I,\ -I,\ -\frac{X}{2},\ -\bar{Q},\ -\frac{I}{2},\ -\bar{R} \right\},\ \Xi_{22} = 0_{6\times 6},$$

$$\Xi_{33} = \mathrm{diag}\left\{ -\frac{I}{2\lambda_2},\ -I,\ -\frac{I}{2},\ -I,\ -I,\ -\frac{I}{2} \right\},$$

in which $\tilde{A}_{11} = AX + XA^{\mathrm{T}} + C\bar{K} + \bar{K}^{\mathrm{T}}C$, $\tilde{A}_{22} = -(1-\rho)\bar{Q}$, $\tilde{A}_{33} = -2\varepsilon X$. Further, the control gain matrix K is designed by

$$K = \bar{K}X^{-1}. \tag{23}$$

- Zhu Quanxin*, Huang Tingwen, H_∞ control of stochastic networked control systems with time-varying delays: The event-triggered sampling case, *International Journal of Robust and Nonlinear Control* 31(18)(2021)9767-9781.

Proof of Theorem 6

- Define $P = X^{-1}$, $Q = \bar{Q}^{-1}$, and $K = \bar{K}X^{-1}$.
- Apply the Schur complement lemma and the following lemma:

Lemma

For any $n \times n$ matrices U, $X > 0$ and positive scalar $\theta > 0$, the following matrix inequality holds:

$$UX^{-1}U^{\mathrm{T}} \geq \theta(U + U^{\mathrm{T}}) - \theta^2 X.$$

Remark

- In Theorems 4-6, our stability criteria depend on the upper bound of delay τ and the upper bound of derivative of delay ρ. Thus, our results are less conservative and more effective than those delay-independent results.
- According to the definition of t_{i+1} and (19), one can know that the interval of arbitrary neighbouring triggered instants has a positive lower bound, i.e., $t_{i+1} - t_i > 0$ for all $i \in \mathcal{Z}_o^+$, which implies that there is no Zeno behavior.
- Compared with the results obtained in [1]-[4], our results are more general. In fact, the authors in [1]-[4] ignored the effects of delays, noise disturbance and unknown parameters. Furthermore, our results are more easily applied in practice than those given in [1]-[4] since they are given by LMIs (see Theorem 6).

Remark

[1] P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, IEEE Trans. Autom. Control, 52(2007)1680-1685.

[2] Y. Xie, Z. Lin, Event-triggered global stabilization of general linear systems with bounded controls, Automatica, 107(2019)241-254.

[3] P. Tallapragada, N. Chopra, On event triggered tracking for nonlinear systems, IEEE Trans. Autom. Control, 58(2013)2343-2348.

[4] E. Garcia, P. Antsaklis, Model-based event-triggered control for systems with quantization and time-varying network delays, IEEE Trans. Autom. Control, 58(2013)422-434.

Example 1

Let us consider the following Itô stochastic nonlinear delay system with exogenous disturbances:

$$\begin{aligned} dx(t) = [&Ax(t) + Bx(t-1) + f(t, x(t)) \\ &+ g(t, x(t-1)) + Cu(t) + \upsilon(t)]dt \\ &+ \sigma(t, x(t), x(t-1))dW(t), \end{aligned} \tag{24}$$

$$A = \begin{bmatrix} 1.5 & -1 \\ 1.2 & 1.2 \end{bmatrix}, B = \begin{bmatrix} 1.3 & -1 \\ 0.9 & 0.8 \end{bmatrix}, C = \begin{bmatrix} 1.2 \\ 0.8 \end{bmatrix},$$

$$\begin{aligned} f(t, x(t)) &= 0.1(1 + \sin(t))x(t), \\ g(t, x(t-1)) &= 0.1(1 + \cos(t))x(t-1), \\ \sigma(t, x(t), x(t-1)) &= \begin{pmatrix} 0.3x_1(t) & 0.3x_2(t-1) \\ 0.2x_2(t) & 0.3x_1(t-1) \end{pmatrix}, \end{aligned}$$

Example 1

The exogenous disturbance input $v(t) = (v_1(t), v_2(t))^T$ is unknown but bounded. From figures (a) and (b), we know that system (24) with $u(t) = 0$ is unstable even if the exogenous disturbance input is missing.

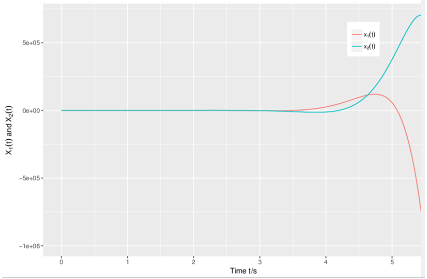

Figure: The sample paths of system (24) with $u(t) = 0$ and $v(t) = 0$

Example 1

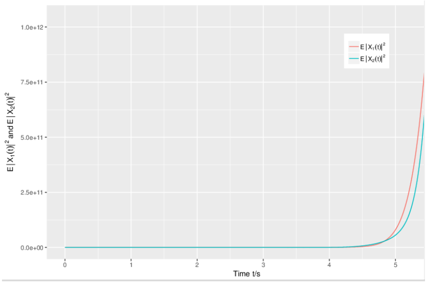

Figure: 2th moment of the solution to system (24) with $u(t) = 0$ and $v(t) = 0$

Example 1

To stabilize system (24), we choose $\eta_1 = 0.1$ and $\eta_2 = 0.1$. By using the Matlab LMI toolbox, we can obtain the following feasible solution for the LMI (10):

$$P = \begin{bmatrix} 28.2232 & 12.7276 \\ 12.7276 & 5.7997 \end{bmatrix}, \quad Q = \begin{bmatrix} 140.9209 & 74.3561 \\ 74.3561 & 41.6116 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 112.3792 & 65.5420 \\ 65.5420 & 41.7449 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 114.1626 & 67.8160 \\ 67.8160 & 44.2443 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 180.2896 & 23.1727 \\ 23.1727 & 141.1538 \end{bmatrix}, \quad G_4 = \begin{bmatrix} 332.8759 & 216.0742 \\ 216.0742 & 152.8354 \end{bmatrix}$$

$$Y = \begin{bmatrix} -191.7285 & -127.8857 \end{bmatrix}.$$

Thus, from Theorem 2 we can design the feedback gain matrix K as

$$K = YP^{-1} = \begin{bmatrix} 304.4865 & -690.2565 \end{bmatrix}.$$

Example 1

Choose the unknown exogenous disturbance $v(t)$ satisfying $v_1(t) = v_0[0.5 + \cos(t)]$ and $v_2(t) = v_0[-0.5 + \sin(t)]$, where v_0 is a sequence of random generator numbers obeying $N(-0.4, 0.4)$ and the initial values is a sequence of random numbers of $U(-1, 1)$. Furthermore, figure (c) shows that the control system is input-to-state practically exponentially mean-square stable.

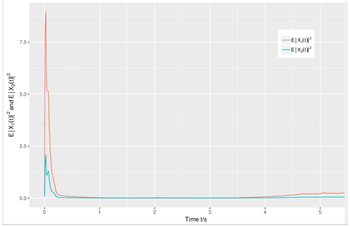

Figure: 2th moment of the solution to system (24) with control and the exogenous disturbance

Example 2

Let us consider the following stochastic networked control delay system:

$$\begin{aligned} dx(t) &= [A(t)x(t) + B(t)x(t-\tau(t)) + C(t)u(t) + F_\upsilon \upsilon(t)]dt \\ &\quad +[G_1 x(t) + G_2 x(t-\tau(t))]dW(t), \\ y(t) &= Dx(t) + Eu(t). \end{aligned} \tag{25}$$

The parameters of system (25) are given as follows:

$$A = \begin{bmatrix} -1.2 & 0 \\ 0 & -1.3 \end{bmatrix}, \; B = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.2 \end{bmatrix}, G_1 = \begin{bmatrix} -0.2 & -0.1 \\ 0.2 & 0.3 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -0.1 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, D = \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & 0.2 \end{bmatrix},$$

Example 2

$$E = \begin{bmatrix} 0.02 & 0 \\ 0 & -0.2 \end{bmatrix}, F_v = \begin{bmatrix} -0.3 & 0.3 \\ 0.1 & 0.4 \end{bmatrix},$$
$$[\triangle A(t), \triangle B(t), \triangle C(t)] = MF(t)[N_1, N_2, N_3],$$
$$M = \begin{bmatrix} 0.3 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, N_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 0.1 & 0.3 \\ -0.1 & 0.2 \end{bmatrix}, \ N_3 = \begin{bmatrix} 0.1 & -0.4 \\ 0.1 & 0.1 \end{bmatrix},$$
$$F(t) = \sin(3\pi t), \ \tau(t) = 1 + 0.2\cos(2\pi t),$$

Choose $\lambda_1 = 0.77, \lambda_2 = 0.05, \gamma = 0.56$. By virtue of MATLAB, the corresponding solutions for (22) can be obtained as follows

Example 2

$$X = \begin{bmatrix} 0.9949 & 0.1148 \\ 0.1148 & 1.5947 \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} 1.3288 & 0.5341 \\ 0.5341 & 1.6456 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} -1.3878 & -0.0584 \\ -0.0584 & 0.8411 \end{bmatrix},$$

Then, from Theorem 44, we obtain the following control matrix:

$$K = \begin{bmatrix} -1.4023 & 0.0643 \\ -0.1205 & 0.5361 \end{bmatrix},$$

Therefore, system (25) is robustly exponentially stablizable in mean-square with an H_∞ disturbance attenuation level $\gamma = 0.56$ by using the above matrix K, the event-triggered parameters $\lambda_1 = 0.77$ and $\lambda_2 = 0.05$.

Example 2

The state response of system (25) without/with the control can be found in Figures 1-3.

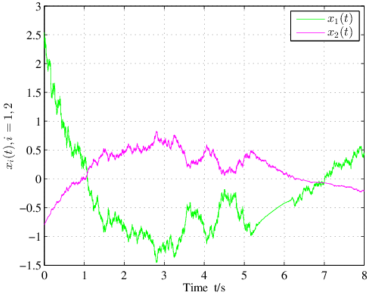

Figure: The state response of system (25) without control under 2-D case.

Example 2

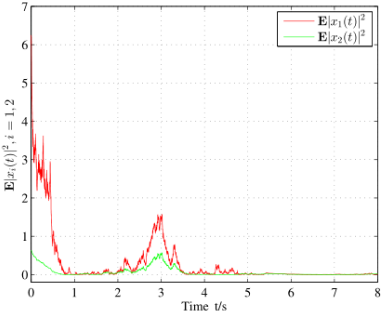

Figure: The state response of system (25) with control under 2-D case.

Example 2

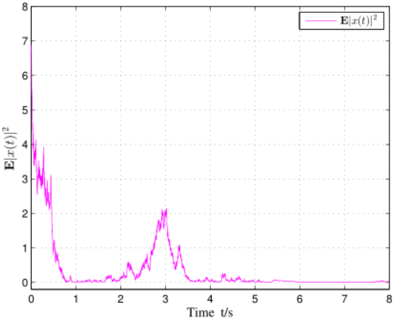

Figure: The time evolution for $\mathbb{E}|x(t)|^2$.

Example 3

The parameters of system (25) with 3-D case are given as follows:

$$A = \begin{bmatrix} -1.4 & 0.2 & 0.2 \\ 0.3 & -1.2 & -0.2 \\ 0.1 & 0.2 & -1.2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4 & -0.2 & 0.3 \\ -0.3 & -0.2 & 0.3 \\ -0.3 & 0.2 & -0.6 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ -0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & -0.2 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -0.2 & -0.1 & 0.2 \\ 0.2 & 0.2 & 0.3 \\ -0.2 & 0.4 & -0.3 \end{bmatrix},$$

Example 3

$$G_2 = \begin{bmatrix} -0.1 & -0.1 & 0.5 \\ -0.2 & -0.3 & 0.2 \\ 0 & 0.5 & -0.5 \end{bmatrix}, \; D^{\mathrm{T}} = \begin{bmatrix} -0.2 \\ 0.3 \\ -0.2 \end{bmatrix},$$

$$E^{\mathrm{T}} = \begin{bmatrix} -0.3 \\ 0.1 \\ 0.2 \end{bmatrix}, \; F_v = \begin{bmatrix} 0.3 \\ -0.3 \\ 0.2 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.1 \end{bmatrix}, \; N_1 = [0.3\ 0.2\ 0.1],$$

$$N_2 = [0.1\ 0.5\ 0.7], \; N_3 = [0.6\ 0.2\ 0.1],$$

$$F(t) = \sin(\pi t), \; \tau(t) = 0.8 + 0.15\cos(\pi t).$$

Choose $\lambda_1 = 2.45, \lambda_2 = 0.01, \gamma = 2.35$.

Example 3

By virtue of MATLAB, the corresponding solutions for (22) can be obtained as follows

$$X = \begin{bmatrix} 1.3652 & -0.4469 & -0.0991 \\ -0.4469 & 1.5488 & 0.2809 \\ -0.0991 & 0.2809 & 0.6286 \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} 1.1501 & -0.2983 & 0.0299 \\ -0.2983 & 1.3472 & 0.2769 \\ 0.0299 & 0.2769 & 0.4277 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} -0.2796 & 0.1296 & 0.0754 \\ 0.1296 & -0.6256 & -1.3884 \\ 0.0754 & -1.3884 & 0.8779 \end{bmatrix},$$

Example 3

Then, from Theorem 44, we obtain the following control matrix:

$$K = \begin{bmatrix} -0.1947 & 0.0123 & 0.0838 \\ -0.0734 & -0.0244 & -2.2094 \\ -0.2347 & -1.3174 & 1.9483 \end{bmatrix}.$$

Therefore, system (25) is robustly exponentially stablizable in mean-square with an H_∞ disturbance attenuation level $\gamma = 2.35$ by using the above matrix K, the event-triggered parameters $\lambda_1 = 2.45$ and $\lambda_2 = 0.01$. The state response of system (25) without/with the control can be found in Figures 4 and 5, respectively.

Example 3

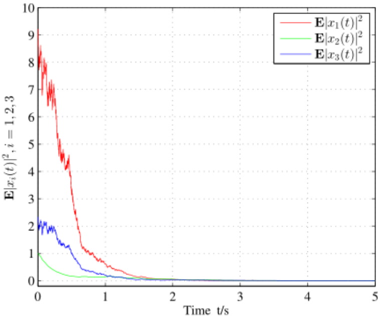

Figure: The state response of system (25) with control under 3-D case.

Example 3

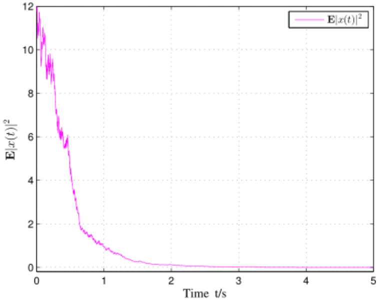

Figure: The state response of system (25) with control under 3-D case.

Our partial publications on this topic (I)

- *Zhu Quanxin, Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control, *IEEE Transactions on Automatic Control*, 64(9)(2019) 3764-3771.
- Yang Xuetao, Wang Hua, *Zhu Quanxin, Event-triggered predictive control of nonlinear stochastic systems with output delay, Automatica 140 (2022) 110230.
- Ding Kui, *Zhu Quanxin, Intermittent static output feedback control for stochastic delayed-switched positive systems with only partially measurable information, *IEEE Transactions on Automatic Control*, (2023) DOI: 10.1109/TAC.2023.3293012.
- *Zhu Quanxin, Huang Tingwen, H_∞ control of stochastic networked control systems with time-varying delays: The event-triggered sampling case, International Journal of Robust and Nonlinear Control 31(18)(2021)9767-9781.

Our partial publications on this topic (II)

- Yang Xuetao, *Zhu Quanxin, Stabilization of stochastic retarded systems based on sampled-data feedback control, IEEE Transactions on Systems, Man, and Cybernetics: Systems 51(2021) 5895-5904.
- Xie Wenjuan, *Zhu Quanxin, Self-triggered state-feedback control for stochastic nonlinear systems with Markovian switching, IEEE Transactions on Systems, Man, and Cybernetics: Systems 50(9)(2020) 3200-3209.
- Ding Kui, *Zhu Quanxin, Fuzzy intermittent extended dissipative control for delayed distributed parameter systems with stochastic disturbance: A spatial point sampling approach, IEEE Transactions on Fuzzy Systems 6(30)(2022)1734-1749.
- *Zhu Quanxin, Zhang Qiuyan, pth moment exponential stabilisation of hybrid stochastic differential equations by feedback controls based on discretetime state observations with a time delay, IET Control Theory Appl., 11(2017)1992-2003.
- Xie Wenjuan, *Zhu Quanxin, Stability of discrete-time stochastic nonlinear systems with event-triggered state-feedback control, Physics A, 547(2020)123823.

Acknowledgements

- Thank you for your attention!!!

Prof. Dr. Quanxin Zhu

Hunan Normal University

Email: zqx22@126.com